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A General Duality Scheme for Nonconvex Minimization Problems with a Strict Inequality Constraint

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Abstract. We establish a duality formula for the problem

$$\text{Minimize } f(x) + g(x) \quad \text{for} \quad h(x) + k(x) < 0$$

where g, k are extended-real-valued convex functions and f, h belong to the class of functions that can be written as the lower envelope of an arbitrary family of convex functions. Applications in d.c. and Lipschitzian optimization are given.

Key words: Strict inequality constraint, Duality, d.c. Optimization, Lipschitzian optimization

1. Introduction

A duality theorem has recently been obtained concerning the minimization of the difference of two convex functions (d.c. function) over a strict inequality d.c. constraint [5, Theorem 3.1, Proposition 3.1]. In this paper we address ourselves to the same problem in the larger class of functions that can be written as the sum of an extended-real-valued convex function and a lower envelope of continuous convex functions.

More precisely, let X be a topological vector space, $g, k : X \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$ two extended-real-valued convex functions, and let $(f_i)_{i \in I}$, $(h_j)_{j \in J}$ be two arbitrary families of convex functions; denoting by $f := \inf_{i \in I} f_i$ (resp. $h := \inf_{j \in J} h_j$) the lower envelope of $(f_i)_{i \in I}$ (resp. $(h_j)_{j \in J}$), we are concerned with the problem

$$(\mathcal{P}) : \quad \text{minimize } f(x) + g(x) \quad \text{for} \quad h(x) + k(x) < 0.$$

It appears that this class of problems covers a great variety of situations including convex programming, d.c. programming, mixed d.c. Lipschitz programming, and minimization problems involving the sum of a convex function and an upper semicontinuous function, etc. Moreover, the fact that the convex functions g and k

can both take the values $-\infty$ and $+\infty$ gives much flexibility to the frame we have chosen.

Although problem $(\mathcal{P})$ is not convex, it is crucial to observe that the component functions $f_i(i \in I)$, and $h_j(j \in J)$ constitute a hidden convex part in $(\mathcal{P})$. So, the main purpose of the paper is to formulate a dual variational principle for the problem $(\mathcal{P})$ by expressing its value in terms of the Legendre–Fenchel conjugate of the functions $f_i(i \in I), g, h_j(j \in J), k$ only.

2. Some facts and notations on convex duality theory

Throughout this paper (X, Y) will be a pair of locally convex topological real linear spaces paired in separating duality by a bilinear form we denote by $\langle , \rangle$. So, X and Y are supplied with topologies compatible with this duality: each of them can be identified with the space of continuous linear forms on the other. With any extended-real-valued function $f : X \to \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$ is associated its Legendre–Fenchel conjugate f^* which is defined on Y by $f^*(y) = \sup_{x \in X} (\langle x, y\rangle - f(x))$ for any $y \in Y$. We denote by dom $f := \{x \in X : f(x) < +\infty\}$ the domain of f, and, for any real number r, we set $\{f \leqslant r\} = \{x \in X : f(x) \leqslant r\}, \{f < r\} = \{x \in X : f(x) < r\}$. Given a subset A of X we denote by δ_A its indicator function ($\delta_A(x) = 0$ if $x \in A$, $\delta_A(x) = +\infty$ if $x \in X \backslash A$). When dealing with the sum of extended-real-valued functions $f_1, \dots, f_n$ on X we adopt the usual convention of convex analysis

$$(+\infty) + (-\infty) = (-\infty) + (+\infty) = +\infty$$

and the related calculus rules (see [6, 7]).

It is well known that the Legendre–Fenchel conjugate of the sum $\sum_{i=1}^{n} f_i$ is strongly related to the infimal convolution of the Legendre–Fenchel conjugate $f_1^*, \dots, f_n^*$. More precisely, let us recall that the infimal convolution of the extended-real-valued functions $f_1^*, \dots, f_n^*$ is defined by

$$(f_1^* \,\square\, \cdots \,\square\, f_n^*)(y) = \inf \left(f_1^*(y_1) + \cdots + f_n^*(y_n) : y_1, \dots, y_n \in Y, \sum_{i=1}^{n} y_i = y \right). \tag{1}$$

Thus, the inequality

$$(f_1 + \cdots + f_n)^* \leqslant f_1^* \,\square\, \cdots \,\square\, f_n^* \tag{2}$$

is always satisfied.

The Fenchel–Moreau–Rockafellar's theorem [1, 6, 9] says that if the extended-real-valued functions $f_1, \dots, f_n$ are convex, if they do not take the value $-\infty$, and if

there exists $\bar{x} \in \text{dom } f_1 \cap \cdots \cap \text{dom } f_n$ such that at least $n-1$ of the f_i are continuous at $\bar{x}$ (3)

then,

$$(f_1 + \cdots + f_n)^* = f_1^* \,\square\, \cdots \,\square\, f_n^* \;, \tag{4}$$

with the infimum in (1) achieved for each $y \in Y$. It has been recently observed [8, Theorem 1] that, under Assumption (3), (4) remains valid if the convex functions take the value $-\infty$. This result will be useful in the sequel. Together with (0), another convention will be used throughout the paper: for any extended-real-valued function $f : X \to \bar{\mathbb{R}}$ we set

$$0f = \delta_{\mathrm{dom}\, f} \;. \tag{5}$$

This amounts to saying that

$$0 \times (+\infty) = +\infty, 0 \times (-\infty) = 0 \;. \tag{6}$$

3. General inequalities

In this section we just assume that $f_i (i \in I), h_j (j \in J), g$ and k are extended-real-valued functions on X; as in Section 1 we set $f := \inf_{i \in I} f_i, h := \inf_{j \in J} h_j$. Let us consider the value $v(\mathcal{P})$ of the problem $(\mathcal{P})$,

$$v(\mathcal{P}) := \inf_{h(x)+k(x)<0} (f(x) + g(x)) \;.$$

Noticing that $h + k = \inf_{j \in J} (h_j + k)$, one has

$$\{h + k < 0\} = \bigcup_{j \in J} \{h_j + k < 0\} \;.$$

Thus,

$$\begin{aligned} v(\mathcal{P}) &= \inf_{j \in J} \inf_{h_j(x)+k(x)<0} (f(x) + g(x)) \\ &= \inf_{j \in J} \inf_{h_j(x)+k(x)<0} \inf_{i \in I} (f_i(x) + g(x)) \;. \end{aligned}$$

Exchanging and gathering the infima one obtains

$$v(\mathcal{P}) = \inf_{(i,j) \in I \times J} \inf_{h_j(x)+k(x)<0} (f_i(x) + g(x)) \;. \tag{7}$$

Let us set

$$v_{i,j} = \inf_{h_j(x)+k(x)<0} (f_i(x) + g(x))$$

for any $(i, j) \in I \times J$.

Observe that $v_{i,j} = +\infty$ whenever the constraint $\{h_j + k < 0\}$ does not meet the domain of the objective function $f_i + g$, that is dom $f_i \cap$ dom g. This situation occurs exactly when dom $f_i \cap$ dom g is included in $\{h_j + k \geqslant 0\}$ or, in other words, when

$$\inf_{x \in \text{dom } f_i \cap \text{ dom } g} (h_j(x) + k(x)) \geqslant 0 .$$

Consequently, in Expression (7) we can restrict the set of indices $(i, j) \in I \times J$ to the subset

$$A = \{(i, j) \in I \times J : \inf_{x \in \text{dom } f_i \cap \text{ dom } g} (h_j(x) + k(x)) < 0\} , \tag{8}$$

so that

$$v(\mathscr{P}) = \inf_{(i,j) \in A} v_{i,j} . \tag{9}$$

Moreover, for any $(i, j) \in I \times J$ one has clearly

$$v_{i,j} \geqslant \inf_{h_j(x)+k(x) \leqslant 0} (f_i(x) + g(x)) = \inf_{x \in X} (f_i(x) + g(x) + \delta_{\{h_j + k \leqslant 0\}}(x)) .$$

Now, taking (5) into account,

$$\delta_{\{h_i + k \leqslant 0\}} = \sup_{\lambda \geqslant 0} (\lambda h_j + \lambda k)$$

for any $j \in J$. This ensures that for all $(i, j) \in I \times J$,

$$\begin{aligned} v_{i,j} &\geqslant \inf_{x \in X} (f_i(x) + g(x) + \sup_{\lambda \geqslant 0} (\lambda h_j(x) + \lambda k(x))) \\ &\geqslant \inf_{x \in X} \sup_{\lambda \geqslant 0} (f_i(x) + g(x) + \lambda h_j(x) + \lambda k(x)) . \end{aligned}$$

By the exchange inf-sup principle we obtain

$$v_{i,j} \geqslant \sup_{\lambda \geqslant 0} \inf_{x \in X} (f_i(x) + g(x) + \lambda h_j(x) + \lambda k(x)) ,$$

or

$$v_{i,j} \geqslant \sup_{\lambda \geqslant 0} -(f_i + g + \lambda h_j + \lambda k)^*(0) ,$$

so that by (2)

$$v_{i,j} \geqslant \sup_{\lambda \geqslant 0} -(f_i^* \,\square\, g^* \,\square\, (\lambda h_j)^* \,\square\, (\lambda k)^*)(0)$$

that is

$$v_{i,j} \geqslant \sup_{\lambda \geqslant 0} \sup_{\sum_{\ell=1}^4 y_\ell = 0} -(f_i^*(y_1) + g^*(y_2) + (\lambda h_j)^*(y_3) + (\lambda k)^*(y_4))$$

for any $(i, j) \in I \times J$.

Now one can state the announced general inequalities:

THEOREM 3.1. *Whatever the extended-real-valued functions* $f_i (i \in I), h_j (j \in J), f, k$ *may be, one always have the inequalities*

$$v(\mathcal{P}) \geqslant \inf_{(i,j)\in A} \sup_{\lambda \geqslant 0} -(f_i + g + \lambda h_j + \lambda k)^*(0)$$

$$\geqslant \inf_{(i,j)\in A} \sup_{\lambda \geqslant 0} \sup_{\sum_{\ell=1}^4 y_\ell = 0} -(f_i^*(y_1) + g^*(y_2) + (\lambda h_j)^*(y_3) + (\lambda k)^*(y_4)),$$

with A as in (8).

4. Strong duality formulas

From now on the extended-real-valued functions g, k, f_i and h_j for all $(i, j) \in I \times J$ will be convex. The following lemma established in [5] by using the inf-sup theorem of Moreau is of particular importance for our purpose; it heavily involves the conventions (0), (6).

LEMMA 4.1 [5, Lemma 3.1]. *Let* p *and* q *be two extended-real-valued convex functions on* X *such that*

$$\text{dom } p \cap \{q < 0\} \neq \emptyset .$$

Then,

$$\inf_{q(x)<0} p(x) = \inf_{q(x)\leqslant 0} p(x) = \max_{\lambda \geqslant 0} \inf_{x\in X} (p(x) + \lambda q(x)) .$$

Applying this lemma one has

$$v_{i,j} = \max_{\lambda \geqslant 0} \inf_{x\in X} (f_i(x) + g(x) + \lambda h_j(x) + \lambda k(x))$$

for all $(i, j) \in A$, and we can state:

THEOREM 4.2. $v(\mathcal{P}) = \inf_{(i,j)\in A} \max_{\lambda \geqslant 0} \inf_{x\in X} (f_i(x) + g(x) + \lambda h_j(x) + \lambda k(x))$.

To go farther one needs additional assumptions.

THEOREM 4.3. *Assume that the convex functions* $f_i (i \in I)$ *and* $h_j (j \in J)$ *are either finite valued and continuous or identically equal to* $-\infty$ *and the condition*

$$\text{there exists } \bar{x} \in \text{dom } g \cap \text{dom } k \text{ s.t. } g \text{ or } k \text{ is continuous at } \bar{x} \tag{10}$$

is satisfied. Then,

$$\left.\begin{array}{l} v(\mathcal{P}) = \inf\limits_{(i,j)\in I\times J(g,k)} \max\limits_{\lambda \geqslant 0} \max\limits_{\substack{y_1,\dots,y_4\in Y \\ y_1+\dots+y_4=0}} -(f_i^*(y_1) + g^*(y_2) + (\lambda h_j)^*(y_3) + (\lambda k)^*(y_4)) , \\ \text{with } J(g,k) = \{j \in J : (h_j^* \Box k^* \Box \delta^*_{\text{dom } g})(0) > 0\} . \end{array}\right\} \tag{11}$$

Proof. We first observe that $\inf_{x\in X}\ (f_i(x)+g(x)+\lambda h_j(x)+\lambda k(x))$ is nothing but $-(f_i+g+\lambda h_j+\lambda k)^*(0)$; taking (10) into account we are in a position to apply the formula (4). It comes out as

$$\begin{aligned}&\inf_{x\in X}\ (f_i(x)+g(x)+\lambda h_j(x)+\lambda k(x))\\&\quad=\max_{\substack{y_1,\cdots,y_4\in Y\\ y_1+y_2+y_3+y_4=0}}\ -(f_i^*(y_1)+g^*(y_2)+(\lambda h_j)^*(y_3)+(\lambda k)^*(y_4))\ .\end{aligned}$$

On the other hand, the set A defined in (8) coincides with $I\times\{j\in J:(h_j+k+\delta_{\mathrm{dom}\,g})^*(0)>0\}$; by (4) and (10) we then have $A=I\times J(g,k)$, and (9) entails (11). □

REMARK . Taking $J=\{1\}$, $h_1=h=0$, and $k=-1$, problem $(\mathscr{P})$ becomes an unconstrained problem:

$$\text{minimize } f(x)+g(x)\quad\text{for}\quad x\in X\ .$$

Assuming that dom $g\neq\emptyset$ we get $J(g,k)=\{1\}$; it then easily follows from (11) that

$$\inf_{x\in X}\ (f(x)+g(x))=\inf_{i\in I}\max_{y\in Y}\ -(f_i^*(y)+g^*(-y))\ .$$

There is another way to obtain a duality formula. Indeed, observe that for any $(i,j)\in A$ one has (see Lemma 4.1)

$$v_{i,j}=\inf_{h_j(x)+k(x)<0}\ (f_i(x)+g(x))=\inf_{h_j(x)+k(x)\leqslant 0}\ (f_i(x)+g(x))\ .$$

It follows that

$$v_{i,j}=((f_i+g)\,\Box\,\delta_{-\{h_j+k\leqslant 0\}})(0),\quad\text{for all}\quad (i,j)\in A\ .$$

Assuming that all the above functions f_i, g, h_j, k coincide with their biconjugate (i.e. belong to $\Gamma(X)$), it is possible to obtain, under additional assumptions, the relation

$$v_{i,j}=((f_i^*\,\Box\,g^*)+\delta^*_{-\{h_j+k\leqslant 0\}})^*(0)\ . \tag{12}$$

More precisely, let us assume that g^* is finite valued and continuous for the Mackey topology $\tau(Y,X)$. Then (see the proof of [6, Prop. 9.b]), $(f_i+g)^*=f_i^*\,\Box\,g^*$ is finite valued (because dom f_i meets dom g) and $\tau(Y,X)$-continuous. By (4) we then have,

$$\begin{aligned}((f_i^*\,\Box\,g^*)+\delta^*_{-\{h_j+k\leqslant 0\}})^*&=(f_i^*\,\Box\,g^*)^*\,\Box\,\delta^{**}_{-\{h_j+k\leqslant 0\}}\\&=(f_i+g)\,\Box\,\delta_{-\{h_j+k\leqslant 0\}}\ ,\end{aligned}$$

so that, under the above assumptions, (12) holds.

Now, by applying Lemma 4.1, observe that

$$-\delta^*_{-\{h_j+k\leqslant 0\}}(y) = \inf_{h_j(x)+k(x)\leqslant 0} \langle x, y\rangle = \max_{\lambda\geqslant 0} -(\lambda h_j + \lambda k)^*(-y) \tag{13}$$

for all $j \in J$ such that $\{h_j + k < 0\} \neq \emptyset$, and all $y \in Y$. Assuming that the functions h_j are continuous and finite valued or identically $-\infty$, it follows from (4) and (13) that

$$\delta^*_{-\{h_j+k\leqslant 0\}}(y) = \min_{\lambda\geqslant 0} [(\lambda h_j)^* \square (\lambda k)^*(-y)] . \tag{14}$$

We are now in a position to state the following result:

THEOREM 4.4. *Assume that $f_i (i \in I)$, g, k belong to $\Gamma(X)$ with g^* finite-valued and $\tau(Y, X)$-continuous, and that $h_j (j \in J)$ is finite-valued and continuous or identically $-\infty$; then*

$$v(\mathcal{P}) = \inf_{(i,j)\in A} \sup_{\lambda\geqslant 0} \sup_{\substack{y_1,\cdots,y_4\in Y\\ y_1+\cdots+y_4=0}} -\left(f_i^*(y_1) + g^*(y_2) + (\lambda h_j)^*(y_3) + (\lambda k)^*(y_4)\right)$$

with $A = \{(i, j) \in I \times J : \inf_{x\in \mathrm{dom}\, f_i\cap \mathrm{dom}\, g} (h_j(x) + k(x)) < 0\}$.

Proof. It follows from (9), (12) and (14) that

$$\begin{aligned} v(\mathcal{P}) &= \inf_{(i,j)\in A} \sup_{y\in Y} \max_{\lambda\geqslant 0} -((f_i^* \square g^*)(y) + ((\lambda h_j)^* \square (\lambda k)^*)(-y)) \\ &= \inf_{(i,j)\in A} \sup_{\lambda\geqslant 0} -((f_i^* \square g^*) \square ((\lambda h_j)^* \square (\lambda k)^*))(0) , \end{aligned}$$

and the result follows from the associativity of the infimal convolution. □

5. Applications

5.1. DUALITY IN D.C. PROGRAMMING

In this section we extend some recent results of the authors concerning the d.c. program below

$$(\mathcal{P}_1) \ : \ \text{minimize } g_1(x) - g_2(A(x)) \text{ for } k_1(x) - k_2(B(x)) < 0 ,$$

where $A : X \to P$ (resp. $B : X \to R$) is a linear continuous operator from X to another ℓ.c.s. P (resp. R) paired in duality with Q (resp. S), g_1, k_1 are extended-real-valued convex functions on X, $g_2 = g_2^{**} \in \Gamma(P)$) and $k_2 = k_2^{**}$ (i.e. $k_2 \in \Gamma(R)$).

In order to apply the results of Section 4, let us notice that for all $x \in X$

$$-g_2(A(x)) = \inf_{q \in \mathrm{dom}\, g_2^*} (-\langle A(x), q\rangle + g_2^*(q)) \ .$$

Denoting by A^* the transpose of A we then have

$$f := -(g_2 \circ A) = \inf_{q \in \mathrm{dom}\, g_2^*} (-\langle \cdot, A^*(q)\rangle + g_2^*(q)) \ .$$

In the same way,

$$h := -(k_2 \circ B) = \inf_{s \in \mathrm{dom}\, k_2^*} (-\langle \cdot, B^*(s)\rangle + k_2^*(s)) \ .$$

Applying Theorem 4.2 with $g = g_1$, $k = k_1$ and f, h as above we obtain the following result that extends Theorem 3.1 of [5]:

THEOREM 5.1. *Assume that g_1, k_1 are extended-real-valued convex functions on X, $g_2 \in \Gamma(P)$, and $k_2 \in \Gamma(R)$; then,*

$$v(\mathcal{P}_1) = \inf_{(q,s)\in Q\times\Delta} \max_{\lambda \geqslant 0} (g_2^*(q) + \lambda k_2^*(s) - (g_1 + \lambda k_1)^*(A^*(q) + \lambda B^*(s))) \ ,$$

where $\Delta = \{s \in S : k_2^*(s) - (k_1 + \delta_{\mathrm{dom}\, g_1})^*(B^*(s)) < 0\}$.

Proof. We have here $I = \mathrm{dom}\, g_2^*$, $J = \mathrm{dom}\, k_2^*$, and for all $(q,s) \in I \times J$, $f_q = -\langle \cdot, A^*(q)\rangle + g_2^*(q)$, $h_s = -\langle \cdot, B^*(s)\rangle + k_2^*(s)$. It follows easily that the set A defined in (8) coincides with $\mathrm{dom}\, g_2^* \times \{k_2^* - (k_1 + \delta_{\mathrm{dom}\, g_1})^* \circ B^* < 0\}$; the rest of the proof is straightforward. □

One can complete Theorem 5.1 in two directions:

1. Assuming the existence of $\bar{x} \in \mathrm{dom}\, g_1 \cap \mathrm{dom}\, k_1$ where g_1 or k_1 is continuous we have [see (4)] $(g_1 + \lambda k_1)^* = g_1^* \square (\lambda k_1)^*$ for all $\lambda \geqslant 0$, and $(k_1 + \delta_{\mathrm{dom}\, g_1})^* = k_1^* \square \delta^*_{\mathrm{dom}\, g_1}$ with the exactness of the above infimal convolution:

2. Assuming $g_1, k_1 \in \Gamma(X)$ with g_1^* finite-valued and $\tau(Y, X)$-continuous we have by Theorem 4.4:

$$v(\mathcal{P}_1) = \inf_{(q,s)\in Q\times\Delta} \sup_{\lambda \geqslant 0} (g_2^*(q) + \lambda k_2^*(s) - (g_1^* \square (\lambda k_1)^*)(A^*(q) + \lambda B^*(s))) \ ,$$

with $\Delta = \{k_2^* - (k_1 + \delta_{\mathrm{dom}\, g_1})^* \circ B^* < 0\}$, a d.c. constraint.

REMARK . Of course, Theorem 5.1 specializes in various situations. For instance, if $k_1 = 0$ we get

$$\inf_{k_2(B(x))>0} (g_1(x) - g_2(A(x)))$$
$$= \inf_{(q,s)\in Q\times\Delta} \max_{\lambda \geqslant 0} (g_2^*(q) + \lambda k_2^*(s) - g_1^*(A^*(q) + \lambda B^*(s)))$$

with $\Delta = \{k_2^* - \delta^*_{\mathrm{dom}\, g_1} \circ B^* < 0\}$.

If, moreover, $g = 0$ (hence $g_2^* = \delta_{\{0\}}$) we have (extending [4, Theorem 4.1] and [10, Corollary 4.6])

$$\inf_{k_2(B(x))>0} g_1(x) = \inf_{s\in\Delta} \max_{\lambda\geqslant 0} (\lambda k_2^*(s) - g_1^*(\lambda B^*(s))) .$$

5.2. DUALITY FOR MIXED D.C.-LIPSCHITZ PROGRAMS

Assume now that X is a normed space with norm $|| \ ||$; we denote by $|| \ ||_*$ the dual norm of the topological dual Y of X: for any $y \in Y$, $||y||_* = \sup_{||x||\leqslant 1} \langle x, y\rangle$, and by $\mathbb{B}_*$ the closed unit ball of Y with center at the origin. Let f be a Lipschitz function on X with c as Lipschitz constant, $g : X \to \bar{\mathbb{R}}$ an extended-real-valued convex function, and let k_1, k_2, B be as in Section 5.1). We are concerned with the problem

$$(\mathcal{P}_2) : \text{ minimize } f(x) + g(x) \quad \text{for} \quad k_1(x) - k_2(B(x)) < 0 .$$

As f is c-Lipschitz one has

$$f(x) = \inf_{u\in X} (c||x-u|| + f(u)) .$$

Therefore we shall take $I = X$ and for all $u \in I$

$$f_u(x) = c||x-u|| + f(u), \quad x \in X ,$$

which is a convex finite-valued and continuous function on X. As in Section 5.1 we shall take $J = \operatorname{dom} k_2^*$ and, for all $s \in \operatorname{dom} k_2^*$, $h_s = -\langle\cdot, B^*(s)\rangle + k_2^*(s)$ which is either an affine continuous function or identically $-\infty$. We have here

$$A = X \times \Delta, \Delta = \{k_2^* - (k_1 + \delta_{\operatorname{dom} g})^* \circ B^* < 0\} .$$

Applying Theorem 4.2 we obtain

$$v(\mathcal{P}_2) = \inf_{(u,s)\in X\times\Delta} \max_{\lambda\geqslant 0} (f(u) + \lambda k_2^*(s) + \inf_{x\in X} (c||x-u|| + g(x) + \lambda k_1(x) - \lambda\langle x, B^*(s)\rangle)) . \tag{15}$$

Let us introduce the function φ_u defined by $\varphi_u(x) = c||x-u||$, $x \in X$, and observe that

$$-\inf_{x\in X} (c||x-u|| + g(x) + \lambda k_1(x) - \lambda\langle x, B^*(s)\rangle) = (\varphi_u + g + \lambda k_1)^*(\lambda B^*(s)) .$$

Assuming that

$$\exists \bar{x} \in \operatorname{dom} g \cap \operatorname{dom} k_1 : g \quad \text{or} \quad k_1 \quad \text{is continuous at} \quad \bar{x} , \tag{16}$$

we have (see (4)) $(\varphi_u + g + \lambda k_1)^* = (\varphi_u)^* \square g^* \square (\lambda k_1)^*$, $(k_1 + \delta_{\operatorname{dom} g})^* = k_1^* \square \delta^*_{\operatorname{dom} g}$ with exactness of the infimal convolutions.

As, moreover, $(\varphi_u)^* = \delta_{c\mathbb{B}_*} + \langle u, \cdot\rangle$, we have proved:

THEOREM 5.2. *Let f, g, k_1, k_2, B be as above and assume that (16) holds; then*

$$v(\mathscr{P}_2) = \inf_{(u,s)\in X\times\Delta} \max_{\lambda\geqslant 0} [(f(u) + \lambda k_2^*(s) + \max_{\substack{\|y_1\|_*\leqslant c \\ y_2\in Y}} (\langle u, y_1\rangle - (g^*(y_2) + (\lambda k_1)^*(y_1 - y_2 + \lambda B^*(s)))] ,$$

with $\Delta = \{k_2^* - (k_1^* \square \delta^*_{\mathrm{dom}\, g}) \circ B^* < 0\}$.

Let us describe the formula above for the problem

$$(\mathscr{P}_3): \text{ minimize } f(x) + g(x) \quad \text{for} \quad k_1(x) \leqslant 0$$

where f is c-Lipschitz, and $g, k_1 : X \to \bar{\mathbb{R}}$ convex satisfying (16) and the Slater condition $\{k_1 < 0\} \cap \mathrm{dom}\, g \neq \emptyset$. By [5, Proposition 2.1] we then have $v(\mathscr{P}_3) = \inf_{k_1(x)<0} (f(x) + g(x))$ and, taking $k_2 = 0$, $B = 0$ in Theorem 5.2, we get (since $\Delta = \{0\}$, $k_2^* = \delta_{\{0\}}$, $B^* = 0$),

$$v(\mathscr{P}_3) = \inf_{u\in X} \max_{\lambda\geqslant 0} \left[f(u) + \max_{\substack{\|y_1\|_*\leqslant c \\ y_2\in Y}} (\langle u, y_1\rangle - (g^*(y_2) + (\lambda k_1)^*(y_1 - y_2))) \right] .$$

Assuming, moreover, that $g = 0$ and $\inf_X k_1 < 0$, we derive

$$\inf_{k_1(x)\leqslant 0} f(x) = \inf_{k_1(x)<0} f(x) = \inf_{u\in X} \max_{\lambda\geqslant 0} [f(u) + \max_{\|y_1\|_*\leqslant c} (\langle u, y_1\rangle - (\lambda k_1)^*(y_1))] ,$$

a duality formula for the minimization of a Lipschitz function over a convex inequality constraint (see [3] for the relevance of such problems). Concerning the minimization of a Lipschitz function f subject to a reverse convex inequality constraint we have, taking $g = k_1 = 0$ and $B = id_X$ in (15), and assuming that $k_2 \in \Gamma(X)$, $\inf_X k_2 \leqslant 0$:

$$\inf_{k_2(x)>0} f(x) = \inf_{u\in X} \inf_{\substack{s\in\mathrm{dom}\, k_2^* \\ s\neq 0}} \left(f(u) + \frac{c}{\|s\|_*}(k_2^*(s) - \langle u, s\rangle)_+ \right)$$

where $(k_2^*(s) - \langle u, s\rangle)_+ = \max\, (k_2^*(s) - \langle u, s\rangle, 0)$.

6. Conclusion

Many duality results can be derived from the general scheme we have presented in Section 4; in a supplement of the classes of functions f, h, considered in Section 5, one may also consider the class of upper semicontinuous (u.s.c.) functions. Indeed, every u.s.c. function f on the normed space X which is majorized by $x \longmapsto \bar{c}\|x -$

$\bar{u}|| + \bar{\alpha}, \bar{c} \geqslant 0, \bar{u} \in X, \bar{\alpha} \in \mathbb{R}$, can be written as follows (see for instance [2, Theorem 3]):

$$f(x) = \inf_{c \geqslant \bar{c}} \inf_{u \in X} (c||x - u|| + f(u)) \quad \text{for all} \quad x \in X \ ;$$

in other words, every u.s.c. function f suitably majorized is the lower envelope of a family of convex continuous functions. Consequently, our results can also be applied to the minimization of the sum of a convex function and an u.s.c. function under a d.c. inequality constraint.

References

1. Fenchel, W. (1953), Convex cones, sets and functions, *Multilith Lecture Notes*, University of Princeton, N.J.
2. Hiriart-Urruty, J.-B. and Volle, M. (1996), Enveloppe k-Lipschitzienne d'une fonction, *Revue de Math. Spé.*, mai–juin.
3. Horst, R. and Tuy. H. (1996), *Global Optimization*, Springer, Berlin / New York.
4. Lemaire, B., Duality in reverse convex optimization, to appear in *SIAM. J. Opt.*
5. Lemaire, B. and Volle, M. (1996), Duality in d.c. programming, *Proceedings of the 5th International Symposium on Generalized Convexity*, C.I.R.M. Luminy, June, 1996, to appear (Kluwer).
6. Moreau, J.-J. (1996), *Fonctionnelles convexes*, Collège de France.
7. Moreau, J.-J. (1970), Inf-convolution, sous-additivité, Convexité des Fonctions Numériques, *Journal de Mathématiques Pures et Appliquées* 49: 109–154.
8. Moussaoui, M. and Volle, M. (1996), Sur la quasicontinuité et les Fonctions Unies en Dualité Convexe, *Comptes-Rendus de l'Académie des Sciences Paris* 322 (I): 839–844.
9. Rockafellar, R.T. (1966), Extension of Fenchel's duality theorem for convex functions, *Duke Mathematics Journal* 33: 81–90.
10. Volle, M. (1997), Quasiconvex duality for the max of two functions, in R. Horst, E. Sachs and R. Tichatschke (eds.), *Lecture Notes in Economics and Mathematical Systems*, No. 452, Springer, Berlin / New York, pp. 365–379.